Cosmological Model of the Universe within the framework of $f(Q)$ gravity theory

Dissertation submitted to

Visvesvaraya National Institute of Technology, Nagpur

in partial fulfillment of the requirement for the award of the degree

Master of Science

in

Mathematics

by

Jishan Alam
(MS23MTH015)

under the supervision of

Dr. G. P. Singh

Department of Mathematics
Visvesvaraya National Institute of Technology
Nagpur - 440010 (India)
May 2025

Cosmological Model of the Universe within the framework of $f(Q)$ gravity theory

Dissertation submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfillment of the requirement for the award of the degree

Master of Science

in

Mathematics

by

Jishan Alam
(MS23MTH015)

under the supervision of

Dr. G. P. Singh

Department of Mathematics
Visvesvaraya National Institute of Technology
Nagpur - 440 010 (India)
May 2025

©Visvesvaraya National Institute of Technology (VNIT) 2025

Department of Mathematics
Visvesvaraya National Institute of Technology, Nagpur

Declaration

I, Jishan Alam, hereby declare that this dissertation titled“Cosmological Model of the Universe within the framework of $f(Q)$ gravity theory” is carried out by me at the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. In this dissertation, i have review two research papers related to accelerated expansion in the $f(Q)$ gravity theory. The review work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other institution/University.

Jishan Alam

Date:

Certificate

This is to certify that the dissertation entitled “Cosmological Model of the Universe within the framework of $f(Q)$ gravity theory” is a bonafide workdone by **Mr Jishan Alam** (Enrollment No. MS23MTH015). The project work has been completed under my guidance and supervision in a satisfactory manner. This work is a part of the requirement for the award of **Master of Science in Mathematics** from Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur. This dissertation fulfills the requirements relating the nature and standard of work for the award of Master of Science degree.

Dr. G. P. Singh
Professor
Dept. of Mathematics,VNIT Nagpur

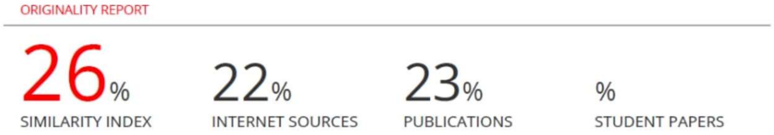
Head, Department of Mathematics
VNIT, Nagpur
Date:

Department of Mathematics

Visvesvaraya National Institute of Technology, Nagpur

Non-Plagiarism Certificate

Certified that M.Sc. Dissertation titled "Cosmological Model of the Universe within the framework of $f(Q)$ gravity theory" submitted by Jishan Alam, Enrollment No. MS23MTH015, ID No. 30507, has been checked for plagiarism using TURNITIN software and the overlap Found within the specified range. Below is an overview of the report.



ORIGINALITY REPORT

26%	22%	23%	%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

Signature:

Student Name: Jishan Alam

Enrollment No: MS23MTH015

ID No: 30507

Dr. G. P. Singh
Professor
Department of Mathematics
VNIT, Nagpur

Acknowledgements

It is of great pleasure and proud privilege to express my deep sense of gratitude to my supervisor **Dr. G. P. Singh**, Professor, Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur for his eminent guidance and generous encouragement throughout my project work. Without his active guidance, it would not have been possible to complete the work. His valuable guidance, not only on the academic front but also on the social front, is gratefully acknowledged.

I convey my deep sense of gratitude to My RPC members Dr. P. Roul and Dr. Jyoti Singh, Department of Mathematics. I am also grateful to all the faculties of the department for their invaluable cooperation throughout my stay at VNIT. My profound gratitude is extended to Mr. Romanshu Garg and Mr. Khomesh Patle, Research Scholar in the Department of Mathematics, VNIT Nagpur for their assistance and unwavering help to complete this dissertation.

My heartfelt thanks to all my friends for their constant inspiration during my study. I express my sincere thanks to my parents and other family members for their extended support, blessings, and encouragement.

Finally, I bow down before the Almighty who has made everything possible.

Jishan Alam

Abstract

This dissertation explores the late-time acceleration of the Universe within the framework of a modified gravity theory known as $f(Q)$ gravity, which uses the non-metricity scalar Q as the central geometric quantity to describe gravitational interactions.

In **Chapter 1**, we provide an overview of essential concepts in cosmology, including the Friedmann–Robertson–Walker (FRW) metric, general relativity, and the standard model of cosmology. The chapter also covers key observational features such as redshift, Hubble's law, and the large-scale structure of the Universe. A detailed discussion is presented on the present cosmological status of the Universe, supported by observational data. The chapter concludes with an introduction to $f(Q)$ gravity as an alternative to general relativity that can potentially account for the observed cosmic acceleration.

Chapter 2 introduces a new parametrization of the Hubble parameter within the context of $f(Q)$ gravity. By adopting a power-law form $f(Q) = Q + mQ^n$, The modified Friedmann equations have been derived. The constrain the model parameters using recent observational datasets, including Hubble parameter measurements, Type Ia Supernovae (Pantheon sample), and Baryon Acoustic Oscillation (BAO) data. The cosmological behavior of parameters such as the deceleration parameter, energy density, isotropic pressure, and equation of state (EoS) parameter is examined. Furthermore, the model is analyzed using statefinder diagnostics and the $Om(z)$ parameter to evaluate its consistency with various dark energy models.

In **Chapter 3**, we propose a novel square-root (SR) parametrization for the EoS parameter of dark energy in $f(Q)$ cosmology. This approach provides a flexible, redshift-dependent description of the EoS and accommodates various dynamical behaviors. Analytical expressions for cosmological observables are derived, and their evolution is studied using observational constraints. The model shows good agreement with current data and reflects quintessence-like behavior, with a consistent transition from a decelerating to an accelerating universe.

Overall, the thesis demonstrates that $f(Q)$ gravity, coupled with appropriate parametrizations, offers a robust framework for modeling the accelerating universe and provides a viable alternative to the standard ΛCDM model.

List of Figures

1.1 The universe might take one of these shapes. 5
1.2 Evolution of the universe depending on curvature and density parameters. 5

2.1 The graphical behavior of the deceleration parameter with z. We take $\alpha = 0.191$, $\beta = 1.21$ 16
2.2 The graphical behavior of the statefinder parameters with z, we take $\alpha = 0.191$, $\beta = 1.21$ 17
2.3 The graphical behavior of the Om diagnostic parameter with z. we take $\alpha = 0.191$, $\beta = 1.21$ 19
2.4 The graphical behavior of the energy density with z and different values for n. . . 20
2.5 The graphical behavior of the isotropic pressure with z and different values for n. 21
2.6 The graphical behavior of the EoS parameter with z and different values for n. . . 22

3.1 The behavior of the deceleration parameter (q) versus redshift (z). 32
3.2 The behavior of the density parameter for ρ_{DE} versus redshift (z). 32
3.3 The behavior of the EoS parameter for DE (ω) versus redshift (z). 33
3.4 The behavior of the Om(z) versus redshift (z). 33

Contents

1 Introduction 1

1.1 General Relativity . . . 1

1.2 Space Time Line Element . . . 2

1.3 Mass Energy Relationship . . . 3

1.4 Friedmann–Robertson–Walker (FRW) model . . . 4

1.5 Cosmology . . . 6

1.6 Redshift . . . 6

1.7 Hubble's Law . . . 7

1.8 Present Status of the Universe at Large Scale . . . 8

1.9 $f(Q)$ Gravity Theory . . . 9

2 A new parametrization of Hubble parameter in $f(Q)$ gravity 11

2.1 Introduction . . . 11

2.2 Formulation of $f(Q)$ Gravity Theory . . . 12

2.3 New Parametrization of the Hubble Parameter . . . 14

2.4 Observational Constraints . . . 14

2.5 Deceleration Parameter and Its Interpretation . . . 15

2.6 Statefinder Analysis . . . 16

2.7 Om Diagnostic . . . 18

2.8 Cosmological $f(Q)$ Model . . . 19

2.9 Conclusion . . . 22

3 Square-root Parametrization of dark energy in $f(Q)$ cosmology 24

3.1 Introduction . . . 24

3.2 $f(Q)$ Cosmology . . . 25

3.3 Square-root (SR) Parametrization for the Equation of State Parameter 28
3.4 Cosmological Parameters 31
3.5 Conclusion 34

Chapter 1

Introduction

1.1 General Relativity

The special theory of relativity is limited to inertial frames, where every physical object moves with the same speed. These inertial frames are not accelerated. Einstein thought that his theory could not be completed until a suitable theory is developed that includes non-inertial systems. So, as a result, he generalized the special theory of relativity by considering natural effect of gravity in the form of general theory of relativity.

General theory of relativity is a geometrical theory of space-time and gravitation. The metric tensor field g_{ij} is a fundamental part of this theory. Principles used in the formulation of general relativity are as follows: The principle of relativity requires all frames to be good frames so that no inertial frame should be chosen as a priority.

1. **Principle of Equivalence**:– The inertial and gravitational forces acting on a body are locally equivalent and non-differential from one another.
2. **Principle of General Covariance**:– It implies that all types of frames are to be equivalent. According to this principle, the nature of the quantities used to form field equations is in Riemann space.
3. **Geodesic Postulate**:– A free particle moves on a geodesic of the corresponding Riemannian space in a gravitational frame.

After considering the above principles, Einstein derived the field equations of general relativity in the form:

$$R_{ik} - \frac{1}{2} R g_{ik} = -\frac{8\pi G}{c^4} T_{ik}. \tag{1.1}$$

Where

- R_{ik} is the Ricci tensor,
- g_{ik} is the fundamental metric tensor,
- G is the Newtonian gravitational constant,
- c is the velocity of light,
- T_{ik} is the energy-momentum tensor due to matter and it takes the form,

$$T_{ik} = (\rho + p) u_i u_k - p g_{ik} \tag{1.2}$$

- R is the Ricci scalar, given by

$$R = g^{ik} R_{ik} = g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} + g^{44} R_{44}. \tag{1.3}$$

This assumption, along with the field equation, forms general relativity. Einstein's field equations represent a very important relationship between the geometry of space-time and matter distribution. In the words of J.A. Wheeler [1]:

> *"Matter tells space how to curve and space tells matter how to move."*

Thus, in general relativity, matter and geometry cannot be separated.

1.2 Space Time Line Element

Let (x, y, z) denote the cartesian coordinate system and t be time measured by an observer O at rest in an inertial frame of reference. Let ds be the space-time interval between two neighboring events in the space-time trajectory of the particle whose coordinates are (x, y, z, t) then

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

It gives the line element as

$$ds^2 = g_{ij}dx^i dx^j. \tag{1.4}$$

where $g_{11} = g_{22} = g_{33} = -1$ and g_{00}=c^2.

1.3 Mass Energy Relationship

Let us consider vector G. The fourth time component is P. The force vector is written in terms of P and G,

$$F^i = [P, G] \tag{1.5}$$

The invariance of the above expression

$$\eta_{mn}u^m F^n = P\frac{dt}{ds} - G\frac{dr}{ds} \tag{1.6}$$

As the left-hand side of the above equation is zero, it vanishes in one inertial frame, so it will be zero in all inertial frames.

$$P = G \cdot V. \tag{1.7}$$

It means $\frac{P}{\gamma}$ is $F \cdot V$ which is the rate of working of force F.

$$\frac{dW}{ds} = \frac{dW}{dt} \cdot \frac{dt}{ds} = \frac{P}{\gamma} \cdot \gamma = P, \tag{1.8}$$

$$\frac{dW}{ds} = \frac{d}{ds}(\gamma \cdot m_0). \tag{1.9}$$

After integrating the above equation, we get

$$W = m + \text{constant}. \tag{1.10}$$

W is interpreted as kinetic energy and it is required to be zero for a particle at rest,

$$W = (m - m_0)c^2. \tag{1.11}$$

Application of this equation is found in the Sun. The core part of the Sun consists of hot plasma in which four protons that combine together and make a helium nucleus by the reaction:

$$4{}^1_0H \rightarrow {}^2_0He + 2\nu + 2e^+ + \gamma \tag{1.12}$$

Neutrinos and positrons are the by-products of this reaction, but the major part of this energy appears in solar radiation. The Sun shines due to $E = mc^2$.

In special theory of relativity, the Doppler effect is the change in wavelength and frequency which is caused due to relative motion of source and observer as

$$v_{\text{observed}} = v_{\text{source}}\sqrt{\frac{1 - \frac{v_s}{c}}{1 + \frac{v_s}{c}}}. \tag{1.13}$$

1.4 Friedmann–Robertson–Walker (FRW) model

Friedmann has developed a mathematical model for expanding universe which was formed on the basis of the modern Big Bang cosmology. The most general form metric which satisfy the cosmological principle (isotropy and homogeneity), consistent with spherical symmetry and allows a uniform expansion of the universe is the FLRW metric and its line element is

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{1.14}$$

Where $a(t)$ is the scale factor which determines the physical size of the universe and its expansion and k is constant, the curvature parameter of the universe having values $k = +1, 0, -1$ for closed, flat and open universes respectively.

The Einstein's field equations (1.1) for the FRW metric (1.14)may be written as

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} \tag{1.15}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G p \tag{1.16}$$

Equations (1.15) takes the form

$$1+\frac{k}{\dot{a}^2}=\Omega. \tag{1.17}$$

Here $\Omega=\frac{\rho}{\rho_c}$, $\rho_c=\frac{3H^2}{8\pi G}$

For closed universe ($k=+1$) the spatial hyper surface has positive curvature $(\Omega>1)$. For flat universe ($k=0$) the spatial hyper surface has curvature equal to zero $(\Omega=1)$. For open universe ($k=-1$) the spatial hyper surface has negative curvature $(\Omega<1)$. A three-dimensional manifold is defined by the universe of one cosmic value. Shape of the manifold does not change. It means that the distance grows larger according to the function $a(t)$ as it is independent of the position called the scale factor.

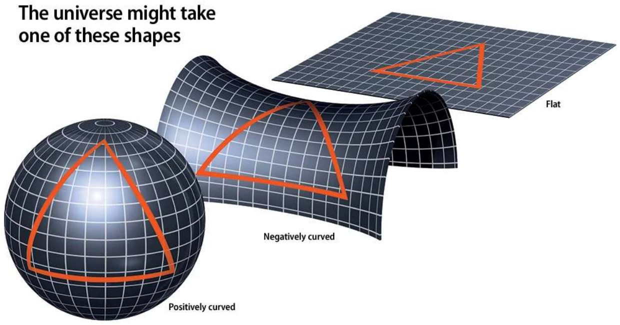

The universe might take one of these shapes
Flat
Negatively curved
Positively curved

Figure 1.1: The universe might take one of these shapes.

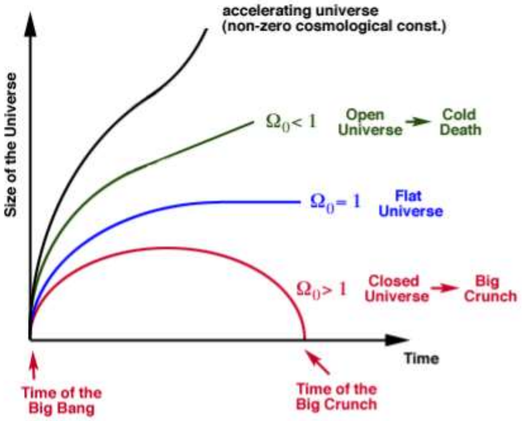


Figure 1.2: Evolution of the universe depending on curvature and density parameters.

1.5 Cosmology

It is a study of the universe as a whole, including how long it came into being its history, its evolution, composition, dynamics and the physical laws which govern it. Our main objective in studying cosmology is to understand the large structure of the universe.

1.7.1 Cosmological Principles

There are three cosmological principles which are the basic assumptions of the universe which are derived from observations. Using these assumptions, a simple model of the universe can be built. As we do not hope that these assumptions are fully satisfied in reality but after comparing the model, we can understand how and why the real one differs.

Homogeneity:- The principle of homogeneity means that the distribution of matter does not change with respect to position.

Isotropy:- The principle of isotropy means there are no special directions in the universe. It says that the universe looks the same in all directions.

Isotropy and homogeneity tell us very different properties of the universe. Isotropy is independent of homogeneity. The universe can be homogeneous but non-isotropic or vice versa.

It is a basic assumption that the physical laws which are discovered on Earth can be applied anywhere in the universe. At large scale universe is isotropic and homogeneous.

1.6 Redshift

Redshift can be described by the relative difference between the observed and emitted wavelength (or frequency of an object). Most of the galaxies show cosmological redshift, which means that detected radiation from the galaxies is shifted to longer wavelengths. It can be measured by the parameter z as

$$z = \frac{\lambda_{\text{observed}} - \lambda_{\text{emitted}}}{\lambda_{\text{emitted}}}.$$

The redshift is due to the Doppler effect. Therefore, if z is positive, it means the galaxy is moving away from us. Receding velocity can be calculated by

$$v = z \times c$$

Where v is the receding velocity and c is the velocity of light.

If the motion of the galaxies were random, then it would be expected that about half of them would recede from us while half of them would approach us. However, Hubble has found that not only most of the galaxies were receding from us, but also the receding velocity is directly proportional to the distance from us.

1.7 Hubble's Law

Hubble's law states that the velocity of the galaxy is directly proportional to its distance.

$$v \propto d,$$

which can be written as

$$v = H_0 \times d,$$

where

- v is the velocity of the galaxy in km/s,
- H_0 is Hubble's constant in km/s/Mpc,
- d is the distance to the galaxy in Mpc.

This law implies that galaxies which are farther away will move away faster. The value of Hubble's constant (H_0) typically ranges from 50 km/s/Mpc to 100 km/s/Mpc. If the value of H_0 is known, we can easily determine the distance of a galaxy using Hubble's Law.

1.9.1 Limitations of the Hubble's Law

Hubble's Law has some limitations, including:

- The observed velocity is influenced by the intrinsic motion of galaxies.
- Galaxies may also be orbiting due to gravitational interactions, affecting the observed velocity.

1.8 Present Status of the Universe at Large Scale

Over the past few decades, substantial observational evidence has reshaped our understanding of the universe on cosmological scales. The most striking discovery is that the universe is not only expanding but doing so at an accelerated rate. This phenomenon, first detected through observations of distant Type Ia supernovae in the late $1990s$, has since been supported by independent probes such as the cosmic microwave background (CMB), baryon acoustic oscillations (BAO), and large scale structure (LSS) surveys [2–4].

At present, the large-scale structure of the universe exhibits a web-like distribution of galaxies, clusters, and voids, forming what is known as the cosmic web. These structures are shaped by the growth of initial density perturbations amplified through gravitational instability over billions of years. On the largest scales above several hundred megaparsecs, the universe appears statistically homogeneous and isotropic, in agreement with the cosmological principle [5].

Current data suggest that the total energy budget of the universe is dominated by components not directly observable. According to the latest results from the Planck satellite, the universe comprises approximately 68% dark energy, 27% dark matter, and only 5% ordinary (baryonic) matter [6]. Dark energy, which drives the accelerated expansion, remains one of the most profound mysteries in cosmology. Its nature is still uncertain, though it is often modeled as a cosmological constant (Λ) or as a dynamic field with a varying equation of state.

The cosmic expansion rate today is described by the Hubble constant H_0. However, there is currently a notable tension between measurements from local distance ladder methods (e.g., using Cepheid variables and supernovae) and those derived from early universe data such as the CMB. Local measurements estimate $H_0 \approx 73\,\text{km/s/Mpc}$, while CMB-based models favor a lower value of $H_0 \approx 67.4\,\text{km/s/Mpc}$ [6,7]. This discrepancy known as the Hubble tension, has led to growing interest in alternative cosmological models, including modified gravity theories like $f(Q)$, $f(R)$, and others.

Furthermore, the large-scale dynamics indicate that the universe transitioned from a matter-dominated decelerating phase to the current dark energy-dominated accelerating phase. The redshift at which this transition occurred is estimated to be around $z \sim 0.6 - 0.8$ [8], depending on the specific model and observational dataset.

In conclusion, the current picture of the universe at large scale is that of an expanding, accelerating cosmos dominated by dark energy. Its structure and evolution continue to challenge and inspire theoretical and observational advancements in cosmology.

1.9 $f(Q)$ Gravity Theory

Modern cosmology, particularly the accelerated expansion of the universe, has spurred considerable interest in modified theories of gravity. Among these, $f(Q)$ gravity emerges as a recent and geometrically novel approach that extends the framework of general relativity by focusing on the non-metricity scalar Q as the fundamental quantity governing gravitational interactions.

In standard GR, gravity is described using the Levi-Civita connection, which is torsion-free and metric-compatible, leading to curvature as the central geometric entity. In contrast, $f(Q)$ gravity belongs to the class of symmetric teleparallel gravity theories, where the connection is curvature-free and torsion-free but allows non-metricity. The non-metricity tensor $Q_{\lambda\mu\nu}$ defined by the covariant derivative of the metric, encapsulates how the length of vectors changes under parallel transport. The scalar Q constructed from contractions of this tensor, represents the gravitational action in this theory [9,10].

The action for $f(Q)$ gravity is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(Q) + \mathscr{L}_m\right], \tag{1.18}$$

where $f(Q)$ is a general function of the non-metricity scalar, g is the determinant of the metric tensor, and $\mathscr{L}_m$ denotes the matter Lagrangian. When $f(Q) = Q$, the theory reduces to the symmetric teleparallel equivalent of general relativity (STEGR), which reproduces the same field equations as GR but via a different geometric interpretation [11].

One of the key motivations for considering $f(Q)$ gravity lies in its ability to explain cosmic acceleration without invoking a dark energy component. This theory introduces new dynamics through the modification of the gravitational action itself. Several studies have demonstrated that $f(Q)$ models can accommodate the current accelerated expansion and even provide alternatives to inflationary scenarios in the early universe [12–14].

Another appealing feature of $f(Q)$ gravity is its second-order field equations, which are free from the instabilities commonly associated with higher-derivative theories like $f(R)$ gravity. Moreover, its formulation avoids issues related to local Lorentz invariance violation found in other modified gravity frameworks such as teleparallel $f(T)$ gravity [15].

Recent cosmological analyses have shown that specific functional forms of $f(Q)$, such as power-law or exponential models, can fit observational data from Type Ia supernovae, baryon acoustic oscillations (BAO), and cosmic microwave background (CMB) observations effectively. These findings support the viability of $f(Q)$ gravity as a promising alternative to ΛCDM and have led to its growing relevance in contemporary gravitational research [16, 17].

In summary, $f(Q)$ gravity presents a geometrically elegant and dynamically rich modification of general relativity. By utilizing the non-metricity scalar instead of curvature or torsion, it offers new insights into cosmic evolution and has the potential to resolve outstanding issues in standard cosmology.

Equations (1.15) and (1.16) are also known as friedmann equation provides the link between the scale factor ($a(t)$) of the universe and the energy density $\rho(t)$ of the matter content in the universe . The acceleration equation will be obtained from the field equation as

$$\frac{\ddot{a}}{a} = -\frac{k^2}{6}(\rho + 3p), \tag{1.19}$$

This acceleration equation shows the condition for which the universe will begin to accelerate,i.e., for

$$\ddot{a} > 0 \implies (\rho + 3p) < 0 \tag{1.20}$$

Chapter 2

A new parametrization of Hubble parameter in $f(Q)$ gravity

2.1 Introduction

In this chapter, we have revisited the research work referred in [24] to investigate the dynamics of an accelerating Universe. We focus on one of the most well-known models in the literature for the function $f(Q)$, defined as $f(Q) = Q + mQ^n$, where m and n are free parameters. The optimal values of these parameters are determined using recent observational datasets that are widely adopted in cosmological studies. The datasets include: (1) the Hubble parameter measurements $H(z)$, comprising 57 data points obtained primarily from the differential age method and other techniques; (2) the Pantheon Type Ia supernova sample, which consists of 1048 data points spanning a redshift range of $0.01 < z < 2.26$; and (3) Baryon Acoustic Oscillation (BAO) data consisting of six points. To constrain the proposed cosmological model, we use a combined analysis of these datasets. Parameter estimation is carried out using the Markov Chain Monte Carlo (MCMC) method, implemented via the 'emcee' library

The structure of the chapter is as follows: first, we present a brief overview of $f(Q)$ gravity theory in a spatially flat FLRW universe. Secondly, we describe the new parametrization of the Hubble parameter and obtain exact solutions to the field equations. We also constrain the model parameters using the combined Hubble, Pantheon, and BAO datasets. Thirdly, we examine a specific $f(Q)$ cosmological model and analyze the behavior of important cosmological quantities such as the energy density, pressure, and the equation of state (EoS) parameter. Finally, we summarize our conclusions and key findings.

2.2 Formulation of $f(Q)$ Gravity Theory

We begin by considering the action that governs the dynamics in $f(Q)$ gravity:

$$S = \int \sqrt{-g} d^4x \left[\frac{1}{2} f(Q) + \mathscr{L}_m\right], \tag{2.1}$$

where $f(Q)$ is a general function of the non-metricity scalar Q, g is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathscr{L}_m$ represents the matter Lagrangian density.

To enforce the flat and symmetric connection (i.e., a connection with zero curvature, $R^{\rho}_{\sigma\mu\nu} = 0$), a Lagrange multiplier is introduced into the action [18].

The trace components of the non-metricity tensor $Q_{\gamma\mu\nu}$ are defined as

$$Q_\gamma = Q_{\gamma}{}^{\mu}{}_{\mu}, \quad \tilde{Q}_\gamma = Q^{\mu}{}_{\gamma\mu},$$

and the associated superpotential tensor (also called the non-metricity conjugate) is given by:

$$4P^{\gamma}{}_{\mu\nu} = -Q^{\gamma}{}_{\mu\nu} + 2Q_{(\mu}{}^{\gamma}{}_{\nu)} - Q^{\gamma} g_{\mu\nu} - \tilde{Q}^{\gamma} g_{\mu\nu} - \delta^{\gamma}_{(\mu Q_\nu)}. \tag{2.2}$$

The non-metricity scalar Q can then be written as:

$$Q = -Q_{\gamma\mu\nu} P^{\gamma\mu\nu}. \tag{2.3}$$

By varying the action with respect to the metric tensor $g_{\mu\nu}$, the field equations are obtained as:

$$\frac{2}{\sqrt{-g}} \nabla_\gamma \left(\sqrt{-g} f_Q P^{\gamma}{}_{\mu\nu}\right) \; + \; \frac{1}{2} g_{\mu\nu} f(Q) \; + \; f_Q \left(P_{\nu\rho\sigma} Q_{\mu}{}^{\rho\sigma} - 2p_{\rho\sigma\mu} Q^{\sigma\rho}{}_{\nu}\right) \; = \; -T_{\mu\nu}, \tag{2.4}$$

where $f_Q \equiv \frac{df}{dQ}$, and the energy-momentum tensor $T_{\mu\nu}$ is defined by:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathscr{L}_m)}{\delta g^{\mu\nu}}. \tag{2.5}$$

Variation of the action with respect to the affine connection yields:

$$\nabla^{\mu}\nabla^{\nu}\left(\sqrt{-g}f_{Q}P^{\gamma}{}_{\mu\nu}\right)=0. \tag{2.6}$$

To analyze the cosmological dynamics, we adopt the spatially flat space time Friedmann–Lemaître–Robertson–Walker (FLRW) metric:

$$ds^{2}=-dt^{2}+a^{2}(t)\left(dx^{2}+dy^{2}+dz^{2}\right), \tag{2.7}$$

where $a(t)$ is the scale factor. In the coincident gauge (i.e., zero connection), the connection becomes trivial, and the metric is treated as the only dynamical variable.

For the FLRW geometry, the non-metricity scalar simplifies to:

$$Q=6H^{2}, \tag{2.8}$$

where $H=\dot{a}/a$ is the Hubble parameter.

We model the matter content as a perfect fluid with energy-momentum tensor:

$$T_{\mu\nu}=(\rho+p)u_{\mu}u_{\nu}+pg_{\mu\nu}, \tag{2.9}$$

where ρ and p are the energy density and isotropic pressure, respectively, and $u^{\mu}=(1,0,0,0)$ is the comoving four-velocity.

Substituting into the field equations, we derive the modified friedmann equations:

$$3H^{2}=\frac{1}{2f_{Q}}\left[-\rho+\frac{1}{2}f(Q)\right], \tag{2.10}$$

$$\dot{H}+3H^{2}+\frac{\dot{f}_{Q}}{f_{Q}}H=\frac{1}{2f_{Q}}\left[p+\frac{1}{2}f(Q)\right]. \tag{2.11}$$

From the modified equations, we extract expressions for energy density and pressure:

$$\rho=\frac{1}{2}f(Q)-6H^{2}f_{Q}, \tag{2.12}$$

$$p=\left(\dot{H}+3H^{2}+\frac{\dot{f}_{Q}}{f_{Q}}H\right)2f_{Q}-\frac{1}{2}f(Q). \tag{2.13}$$

In the limiting case where $f(Q) = Q = 6H^2$, the theory reduces to the symmetric teleparallel equivalent of General Relativity (STEGR) [20], and the equations become:

$$3H^2 = \rho, \quad 2\dot{H} + 3H^2 = -p, \tag{2.14}$$

which are the standard Friedmann equations.

2.3 New Parametrization of the Hubble Parameter

The system of field equations discussed earlier comprises only two independent equations for four unknowns—namely, the energy density ρ, pressure p, Hubble parameter H, and the function $f(Q)$. To achieve a complete description of the Universe's evolution and determine cosmological parameters, additional constraints must be introduced.

From a mathematical perspective, solving this underdetermined system necessitates incorporating supplementary relations. In cosmology, a widely adopted technique is to assume model-independent parametrizations of key parameters such as the Hubble parameter $H(z)$, the deceleration parameter $q(z)$, or the equation of state (EoS) parameter $\omega(z)$. This approach is particularly useful in studying the dynamics of dark energy without assuming a specific underlying theory [21].

Proposed Hubble Parameter Form

In this study, we employ a new parametrization of the Hubble parameter, defined as [24]:

$$H(z) = H_0 \left[\alpha(1-z) + \beta(1+z) + \frac{z}{2} \right], \tag{2.15}$$

where H_0 is the present value(at $z = 0$) of the Hubble parameter .

2.4 Observational Constraints

To examine the behavior of the cosmological parameters and evaluate the physical viability of our model, we aim to constrain the parameters α and β using current observational datasets. These include:

- Hubble parameter measurements $H(z)$ across various redshifts,

- Type Ia supernova data from the Pantheon sample,
- Baryon Acoustic Oscillation (BAO) data.

A statistical analysis will be performed to find the best-fit values for the model parameters by minimizing the chi-square function constructed from the combined datasets.

2.5 Deceleration Parameter and Its Interpretation

The deceleration parameter $q(z)$ serves as a crucial indicator of the Universe's expansion phase. It is known that the Universe takes a transition from a decelerating phase ($q > 0$) in the past to the current accelerating phase ($q < 0$). Observations suggest that $q(z)$ lies in the range $-1 < q < 0$ during the present epoch. if $q = 0$ then there is neither an accelerating nor a decelerating phase, and the expansion rate of the universe is constant.

The deceleration parameter is related to the Hubble parameter through the relation:

$$q = -1 + \frac{(1+z)}{H(z)}\frac{dH}{dz}. \tag{2.16}$$

From Equations (2.15) and (2.16), we obtain the $q(z)$ for this model:

$$q(z) = \frac{\alpha + \beta + \alpha(-z) + \beta z - 2}{2(\beta z^2 + z(\alpha+\beta) + 1)}. \tag{2.17}$$

The behavior of the deceleration parameter $q(z)$, computed using the best-fit values of the model parameters is illustrated in figure (2.1). This figure clearly demonstrates that the proposed cosmological model exhibits a transition from a decelerating to an accelerating phase in the expansion history of the Universe.

The redshift at which this transition occurs, denoted by z_t, is found to be approximately $z_t = 0.6857$ based on constraints derived from combined observational data sets, including Hubble parameter measurements, the Pantheon compilation of Type Ia supernovae, and Baryon Acoustic Oscillation (BAO) data. Additionally, the present value of the deceleration parameter at present $(z = 0)$ is $q_0 = -0.3092$. This negative value confirms that the Universe is currently in an accelerated expansion phase.

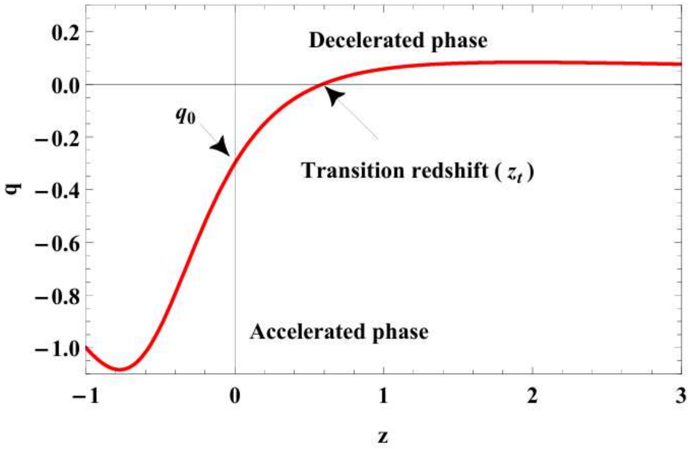


Figure 2.1: The graphical behavior of the deceleration parameter with z. We take $\alpha = 0.191$, $\beta = 1.21$

2.6 Statefinder Analysis

The deceleration parameter q is a valuable tool to characterize the expansion behavior of the universe. It becomes insufficient for distinguishing between various dark energy (DE) models. To address this limitation, a more refined diagnostic tool known as the *Statefinder* pair (r,s) [42] has been proposed, providing deeper insight into the underlying dynamics of cosmological models.

$$r = \frac{\dddot{a}}{aH^3}, \tag{2.18}$$

$$s = \frac{r-1}{3(q-1/2)}, \tag{2.19}$$

where $q \neq 1/2$.

Different DE models are characterized by unique trajectories in the (r,s) plane:

- The ΛCDM model corresponds to the fixed point $(r = 1, s = 0)$,
- The Chaplygin Gas (CG) /phantom scenario model satisfies $r > 1$, $s < 0$,
- Quintessence models typically yield $r < 1$, $s > 0$,
- Holographic dark energy (HDE) model corresponds to the fixed point $(r = 1, s = 2/3)$,
- Standard cold dark matter (SCDM) model corresponds to the fixed point $(r = 1, s = 1)$.

From Eqs 2.15 and 2.18 we obtain,

$$r(z) = \frac{1-\alpha}{\beta z^2 + z(\alpha+\beta)+1}, \tag{2.20}$$

From Eqs 2.17 and 2.20 we get,

$$s(z) = \frac{2(z+1)(\alpha+\beta z)}{3\beta(z^2-1)+\alpha(6z-3)+9}, \tag{2.21}$$

where α and β are the model parameters. Figure 2.2 illustrates the r–s trajectory corresponding to the best-fit parameters derived from the combined datasets, including Hubble parameter data, the Pantheon Type Ia supernova sample, and BAO measurements.The plot reveals that the

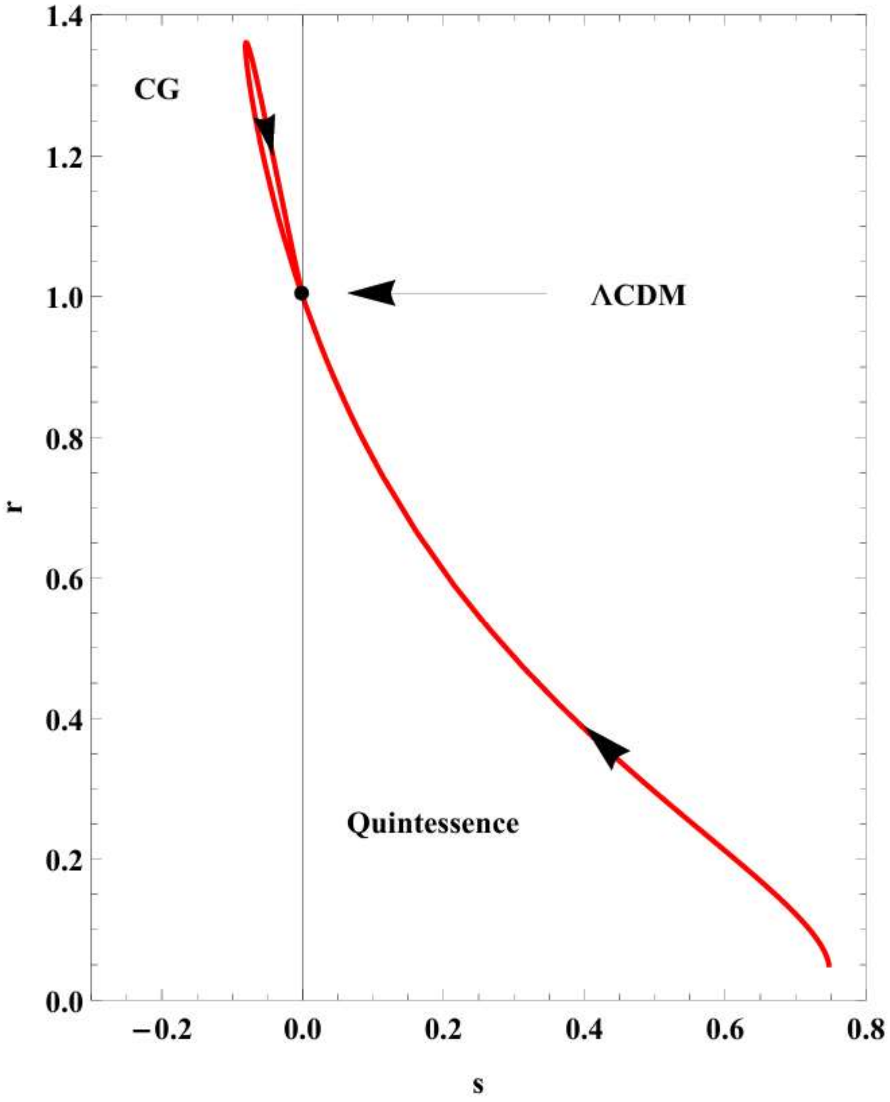


Figure 2.2: The graphical behavior of the statefinder parameters with z, we take $\alpha = 0.191$, $\beta = 1.21$

model begins in a regime resembling quintessence behavior $(r < 1, s > 0)$, transitions through the Chaplygin Gas phase $(r > 1, s < 0)$ at intermediate redshifts, and eventually approaches the ΛCDM fixed point $(r = 1, s = 0)$ at late times. This dynamic evolution showcases the versatility and consistency of the proposed model in explaining various phases of cosmic expansion.

2.7 Om Diagnostic

The Om diagnostic provides a straightforward yet powerful method for differentiating between various dark energy (DE) models, based solely on the Hubble parameter. Unlike other diagnostics that require higher-order derivatives of the scale factor, the Om diagnostic depends only on the first derivative, making it computationally simpler and observationally more accessible.

In the context of a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) Universe, the Om parameter is defined as:

$$Om(z) = \frac{E^2(z) - 1}{(1+z)^3 - 1}, \tag{2.22}$$

where $E(z) = \frac{H(z)}{H_0}$ is the normalized Hubble parameter, and H_0 is its present-day value.

This diagnostic serves as an indicator of the underlying nature of dark energy:

- A negative slope in $Om(z)$ suggests a quintessence-like model,
- A positive slope corresponds to a phantom-like behavior,
- A constant value of $Om(z)$ aligns with the standard ΛCDM model.

From Eqs (2.15) and (2.22), we get Om diagnostic for this model,

$$Om(z) = \frac{\alpha + \beta + \beta z}{z^2 + 3z + 3}, \tag{2.23}$$

where α and β are model parameters. As depicted in figure (2.3), the Om diagnostic curve initially exhibits a negative slope, indicative of quintessence behavior in the early evolution of the Universe. However, at later times, the slope becomes positive, which reflects a transition towards phantom-like dynamics. This dual-phase behavior highlights the flexibility of the proposed model in capturing different expansion histories of the Universe.

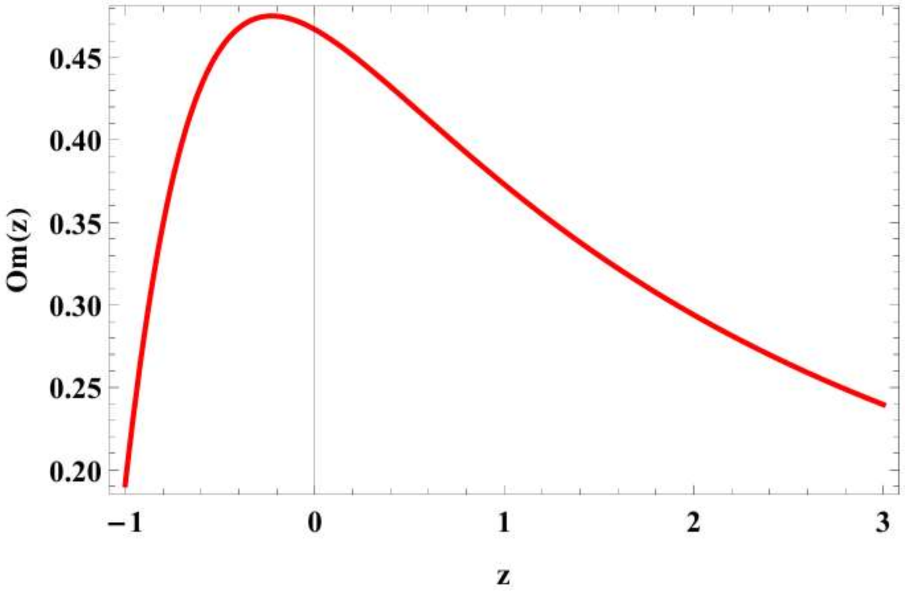


Figure 2.3: The graphical behavior of the Om diagnostic parameter with z. we take $\alpha = 0.191$, $\beta = 1.21$

2.8 Cosmological $f(Q)$ Model

In this section, we explore a cosmological scenario within the framework of $f(Q)$ gravity, making use of the newly introduced parametrization of the Hubble parameter discussed earlier. The analysis is conducted using the constrained values of the model parameters obtained from combined observational datasets including Hubble, Pantheon, and BAO.

To distinguish between different evolutionary phases of the universe, we employ the equation of state (EoS) parameter, defined as:

$$\omega = \frac{p}{\rho}, \tag{2.24}$$

where p represents the pressure and ρ is the energy density.

In the standard cosmological model under general relativity, a constant EoS parameter $\omega = -1$ corresponds to the cosmological constant, which drives the accelerated expansion of the universe. However, other dynamical dark-energy models may admit different ranges:

- Quintessence: $-1 < \omega < -\frac{1}{3}$,
- Phantom: $\omega < -1$,
- Acceleration: Requires $\omega < -\frac{1}{3}$.

To construct our model, we consider a power-law form of the non-metricity scalar function as follows:

$$f(Q) = Q + mQ^n, \tag{2.25}$$

where m and n are free parameters.

Substituting Eq. (2.25) into the modified Friedmann equations (Eqs. (2.12) and (2.13)), the expressions for the energy density and pressure become:

$$\rho = \frac{1}{2}\left[m(-6^n)(2n-1)(H^2)^n - 6H^2\right], \tag{2.26}$$

$$p = \frac{1}{6H^2}\left[2\dot{H}\left(m6^n n(2n-1)(H^2)^n + 6H^2\right) + 3H^2\left(m6^n(2n-1)(H^2)^n + 6H^2\right)\right], \tag{2.27}$$

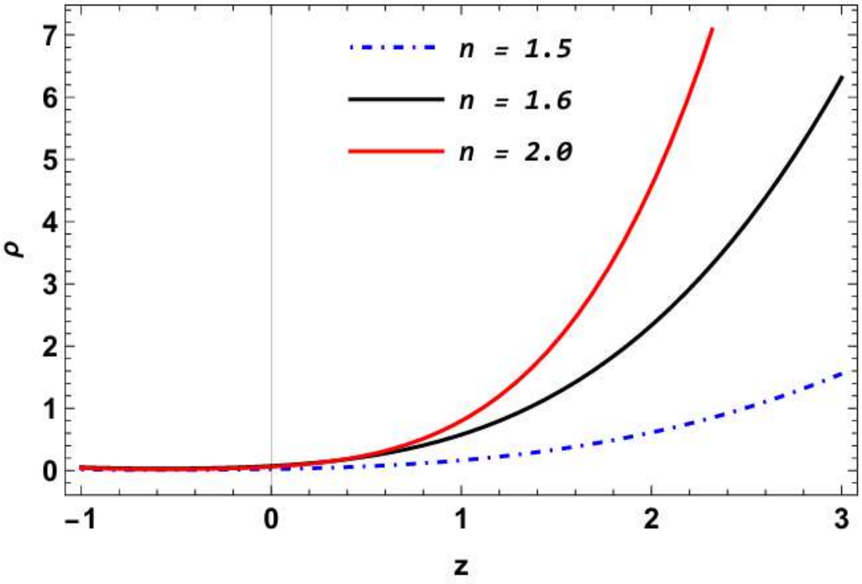


Figure 2.4: The graphical behavior of the energy density with z and different values for n.

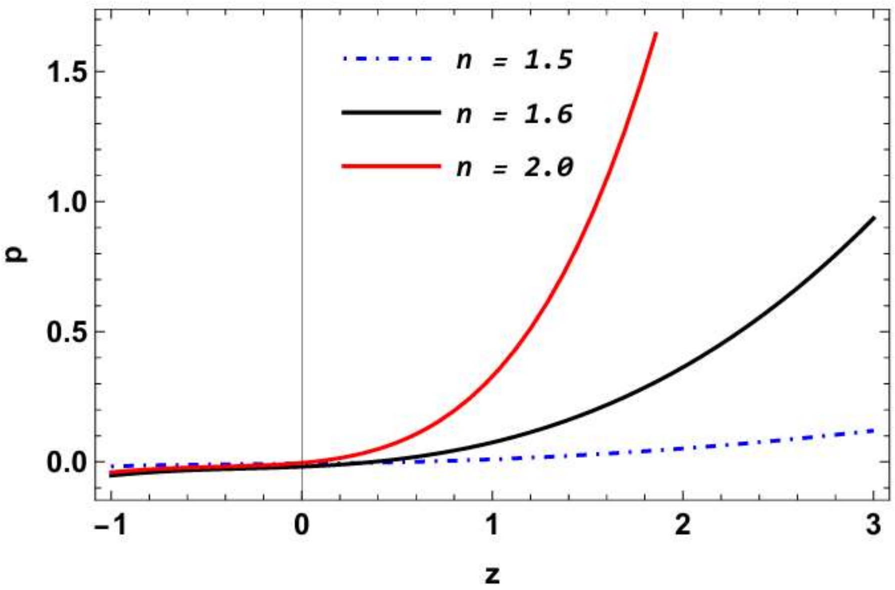


Figure 2.5: The graphical behavior of the isotropic pressure with z and different values for n.

From figure 2.4, it is evident that the energy density of the universe increases with redshift z (or equivalently, decreases with cosmic time t) and remains positive throughout the cosmic evolution for all three considered values of the parameter n. The energy density initiates from a finite positive value in the early Universe and gradually declines towards zero as the Universe expands, which aligns with theoretical expectations.

Figure 2.5 illustrates the variation of pressure with redshift. For all values of n, the pressure is initially positive at high redshift (corresponding to the early universe) and progressively transitions to negative values in the late-time and present-day universe. This negative pressure at late times is consistent with the influence of exotic energy components such as dark energy, which are responsible for the accelerated expansion of the Universe.

In Figure 2.6, the equation of state (EoS) parameter ω is plotted against redshift z using a power-law form for the non-metricity function under varying values of n. The behavior reveals that the model starts in a matter-dominated regime ($\omega = 0$) at early times, evolves through a quintessence-like phase ($-1 < \omega < -\frac{1}{3}$) in the present epoch, and approaches the cold dark matter (CDM) regime ($\omega = -1$) around $z = 1$. Specifically, for $n = 1.5$, the EoS parameter shows clear quintessence behavior at the current epoch, which is also reflected in the evolution of the deceleration parameter q, demonstrating consistent dynamics.

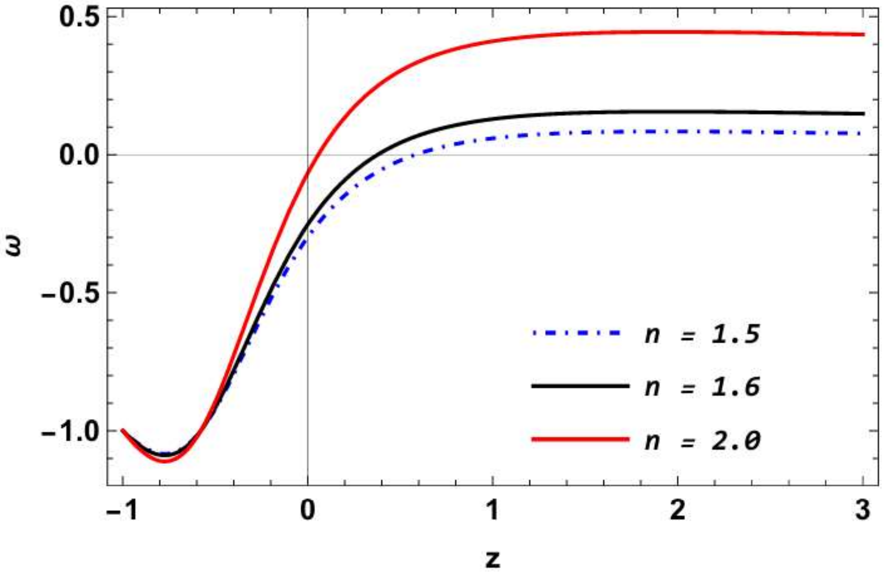


Figure 2.6: The graphical behavior of the EoS parameter with z and different values for n.

2.9 Conclusion

In this work, we have explored the late-time accelerated expansion of the universe within the framework of $f(Q)$ gravity. To derive exact solutions of the field equations under the FLRW metric,a new parametric form of the Hubble parameter has been proposed

$$H(z) = H_0\left[(1-\alpha)+(1+z)(\alpha+\beta z)\right]^{\frac{1}{2}},$$

where α and β are constants constrained using observational datasets including Hubble parameter measurements, the Pantheon compilation, and BAO data. The best-fit values were found to be $\alpha = 0.191^{+0.093}_{-0.093}$ and $\beta = 1.21^{+0.13}_{-0.13}$.

With these constraints, we analyzed the evolution of several cosmological parameters. The deceleration parameter $q(z)$ exhibits a transition from a decelerating to an accelerating phase, with the transition redshift calculated as $z_t = 0.6857$. The current value of the deceleration parameter is $q_0 = -0.3092$, consistent with recent observational data supporting cosmic acceleration.

A statefinder diagnostic was employed to further characterize of the model. The (r,s) trajectory shows that the model initially aligns with quintessence-like behavior ($r < 1$, $s > 0$), transitions to the Chaplygin gas regime ($r > 1$, $s < 0$), and eventually converges to the ΛCDM fixed point ($r = 1$, $s = 0$).

The Om diagnostic was also considered, which depends solely on the Hubble parameter and is useful for differentiating dark energy models. The Om parameter exhibits a negative slope at early times, indicating quintessence-like dynamics. In contrast, a positive slope is observed in the future, suggesting a possible evolution toward phantom behavior.

To examine the dynamical aspects of $f(Q)$ gravity more closely, we assumed a power law form for the function, namely $f(Q) = Q + mQ^n$, where m and n are free parameters. The resulting expressions for energy density and isotropic pressure, derived from this functional form, indicate that energy density remains positive and increases with redshift (decreasing with cosmic time), reflecting the expansion of the universe. Furthermore, the pressure is negative at present and future epochs, which is a necessary condition for cosmic acceleration.

Lastly, the evolution of the Equation of state (EoS) parameter reveals that the model exhibits a quintessence-like nature at the current epoch. Based on these findings, we conclude that the proposed $f(Q)$ model with the new parametrization of the Hubble parameter successfully captures the essential features of the present accelerating universe and is consistent with multiple observational constraints.

Chapter 3

Square-root Parametrization of dark energy in $f(Q)$ cosmology

3.1 Introduction

In this chapter we have revisited the work done in the reference [29]. Over the past few decades, a wide range of astronomical observations including high-redshift Type Ia Supernovae (SNe), Cosmic Microwave Background (CMB) radiation, Baryon Acoustic Oscillations (BAO) and large-scale structure (LSS) surveys have collectively confirmed that the Universe is currently undergoing a phase of accelerated expansion. This unexpected phenomenon poses one of the most significant challenges in modern cosmology. The unknown driver of this acceleration is referred to as dark energy (DE), the precise nature of which remains one of the deepest mysteries in theoretical physics.

The cosmological constant (Λ) is the simplest candidate for DE and is widely employed in the standard ΛCDM model. However, it suffers from well-known theoretical issues such as the fine-tuning problem and the coincidence problem. These concerns have motivated the development of alternative approaches to model DE more accurately.

One common strategy involves the parameterization of the equation of state (EoS) parameter $w(z)$ of DE as a function of redshift or scale factor. Such phenomenological models allow researchers to capture the dynamical behavior of DE without relying on a specific underlying theory of gravity. These models can be constrained directly by observational data, making them a powerful tool in model-independent cosmological analysis.

Another promising route is the modification of Einstein's theory of General Relativity (GR). Among the various modified gravity theories, $f(Q)$ gravity has recently garnered considerable attention. This theory is grounded in the concept of non-metricity, which measures deviations from metric compatibility in the affine connection. Unlike curvature-based GR or torsion-based teleparallel gravity, $f(Q)$ gravity belongs to the symmetric teleparallel framework characterized by zero curvature and torsion. It offers a fresh geometric perspective for explaining cosmic acceleration.

In this chapter, we explore a novel square-root parameterization of the dark energy EoS parameter [29] given by

$$\omega_{SR} = \omega_0 + \omega_1 \frac{z}{\sqrt{z^2+1}},$$

where ω_0 and ω_1 are free parameters. We study this model within the context of the $f(Q)$ gravity framework adopting the functional form $f(Q) = Q + \alpha Q^n$, where α and n are constants. By utilizing the latest observational datasets including 31 Hubble parameter measurements and the Pantheon sample of 1048 SNe, we constrain the model parameters and analyze the evolution of key cosmological quantities. These include the deceleration parameter, the dark energy EoS parameter and the $Om(z)$ diagnostic. Our goal is to evaluate the viability of this model in explaining the current phase of cosmic acceleration and to gain insight into the possible nature of dark energy.

3.2 $f(Q)$ Cosmology

In $f(Q)$ gravity, the covariant derivative of the metric tensor does not vanish, which introduces a new geometric entity known as non-metricity. Mathematically, this is expressed as $Q_{\sigma\mu\nu} = \nabla_\sigma g_{\mu\nu}$. Unlike General Relativity, where gravitational effects arise from spacetime curvature, $f(Q)$ gravity is formulated in a geometry that is curvature and torsion free, relying entirely on non-metricity to describe gravitational interactions.

The action for $f(Q)$ gravity takes the form:

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2} f(Q) + \mathscr{L}_m \right], \tag{3.1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $\mathscr{L}_m$ is the matter Lagrangian density, and $f(Q)$ is a general function of the non-metricity scalar Q.

The traces of the non-metricity tensor are defined as:

$$Q_\sigma = Q_\sigma{}^\mu{}_\mu, \quad \tilde{Q}_\sigma = Q^\mu{}_{\sigma\mu}. \tag{3.2}$$

The non-metricity scalar Q is defined through the superpotential $P^{\sigma\mu\nu}$ as:

$$Q = -Q_{\sigma\mu\nu}P^{\sigma\mu\nu}, \tag{3.3}$$

where the superpotential is given by:

$$4P^\sigma{}_{\mu\nu} = -Q^\sigma{}_{\mu\nu} + 2Q_{(\mu}{}^{\sigma}{}_{\nu)} - Q^\sigma g_{\mu\nu} - \tilde{Q}^\sigma g_{\mu\nu} - \delta^\sigma_{(\mu}Q_{\nu)}. \tag{3.4}$$

Varying the action with respect to the metric yields the field equations:

$$\frac{2}{\sqrt{-g}}\nabla_\sigma\left(\sqrt{-g}f_Q P^\sigma{}_{\mu\nu}\right) + \frac{1}{2}g_{\mu\nu}f(Q) + f_Q\left(P_{\mu\sigma\beta}Q_\nu{}^{\sigma\beta} - 2Q_{\sigma\beta\mu}P^{\sigma\beta}{}_\nu\right) = T_{\mu\nu}, \tag{3.5}$$

where $f_Q = \frac{df}{dQ}$ and $T_{\mu\nu}$ is the energy-momentum tensor for matter given by:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{3.6}$$

assuming a perfect fluid with energy density (ρ), pressure (p) and four-velocity (u_μ) satisfying $u_\mu u^\mu = -1$.

We consider a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric:

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{3.7}$$

where $a(t)$ is the scale factor. For this metric, the non-metricity scalar becomes:

$$Q = 6H^2, \tag{3.8}$$

with $H = \dot{a}/a$ being the Hubble parameter.

The modified Friedmann equations in $f(Q)$ gravity are:

$$3H^2 = \frac{1}{2f_Q}\left(\rho + \frac{f}{2}\right), \tag{3.9}$$

$$\dot{H} + \left(3H + \frac{\dot{f}_Q}{f_Q}\right)H = \frac{1}{2f_Q}\left(-p + \frac{f}{2}\right). \tag{3.10}$$

For the specific case $f(Q) = Q$, these reduce to the standard Friedmann equations of GR. However, if $f(Q) = Q + F(Q)$ then the equations modify to:

$$3H^2 = \rho + \frac{F}{2} - QF_Q, \tag{3.11}$$

$$(2QF_{QQ} + F_Q + 1)\dot{H} + \frac{1}{4}(Q + 2QF_Q - F) = -2p. \tag{3.12}$$

Equations (3.11) and (3.12) can be viewed as the symmetric teleparallel analogues of the standard cosmological equations in GR, with an additional contribution emerging from the non-metricity scalar Q. This contribution behaves similarly to an effective fluid component that can be associated with dark energy, such that $\rho_Q = \rho_{\mathrm{DE}}$ and $p_Q = p_{\mathrm{DE}}$. Hence, the modified Friedmann equations take the form:

$$H^2 = \frac{1}{3}\left(\rho + \rho_{\mathrm{DE}}\right), \tag{3.13}$$

$$2\dot{H} + 3H^2 = -\left(p + p_{\mathrm{DE}}\right), \tag{3.14}$$

Here ρ_{DE} and p_{DE} represent the effective energy density and pressure arising from the non-metricity of spacetime and are defined as follows:

$$\rho_{\mathrm{DE}} = \frac{F}{2} - QF_Q, \tag{3.15}$$

$$p_{\mathrm{DE}} = 2\dot{H}(2QF_{QQ} + F_Q) - \rho_{\mathrm{DE}}. \tag{3.16}$$

Consequently, the EoS parameter for the dark energy component is given by:

$$w_{\mathrm{DE}} = \frac{p_{\mathrm{DE}}}{\rho_{\mathrm{DE}}} = -1 + \frac{4\dot{H}(2QF_{QQ} + F_Q)}{F - 2QF_Q}. \tag{3.17}$$

Finally, the conservation equation for a perfect fluid in this theory is:

$$\dot{\rho} + 3H(1+\omega)\rho = 0, \tag{3.18}$$

which follows from the covariant divergence of the energy-momentum tensor, i.e. $\nabla^{\mu} T_{\mu\nu} = 0$ [27].

3.3 Square-root (SR) Parametrization for the Equation of State Parameter

To investigate the SR parametrization, we consider the functional form $F(Q) = \alpha Q^n$, where α and n are free parameters of the model. This particular choice is inspired by the work in [33]. When $n = 0$, the function reduces to the ΛCDM model, where $\frac{\alpha}{2}$ effectively plays the role of the cosmological constant [30, 31]. For $n = 1$, the model becomes equivalent to symmetric teleparallel gravity with a rescaling of Newton's gravitational constant by a factor of $\alpha + 1$ [32]. It is important to note that deviations from General Relativity (GR) arise depending on the value of the parameter n. Specifically, for $n < 1$, the modifications to the gravitational dynamics become significant in the low-curvature regime, making such models suitable for describing the late-time, dark energy-dominated phase of the Universe. On the other hand, models with $n > 1$ exhibit prominent deviations in high-curvature conditions and are therefore more applicable to the dynamics of the early Universe [33].

Using the power-law form of $f(Q)$, the dark energy density ρ_{DE} and pressure p_{DE} are expressed in terms of the Hubble parameter H as follows:

$$\rho_{\mathrm{DE}} = \alpha 6^n \left(\frac{1}{2} - n\right) H^{2n}, \tag{3.19}$$

$$p_{\mathrm{DE}} = -\alpha 6^{n-1} \left(\frac{1}{2} - n\right) H^{2(n-1)} (3H^2 + 2n\dot{H}). \tag{3.20}$$

The EoS parameter for dark energy becomes:

$$\omega_{\mathrm{DE}} = -1 - \frac{2n}{3}\left(\frac{\dot{H}}{H^2}\right). \tag{3.21}$$

To facilitate cosmological analysis and enable a direct comparison between theoretical predictions and observational data, we introduce the redshift parameter z, defined by the relation

$$1+z = \frac{1}{a},$$

where the scale factor a is normalized such that its present-day value is unity, i.e., $a(t_0) = a_0 = 1$. Consequently, we express cosmological quantities as functions of redshift z instead of cosmic time t. Under this transformation, the time derivative of the Hubble parameter H can be reformulated accordingly.

$$\dot{H} = \frac{dH}{dt} = -(1+z)H(z)\frac{dH(z)}{dz}. \tag{3.22}$$

In our current framework, we have the flexibility to choose an additional parameter, since the system of equations, specifically equations (3.19)-(3.21), contains more unknowns than equations. The dark energy (DE) component, which is central to explaining the accelerated expansion of the Universe, is commonly described using its EoS parameter. This parameter is defined as the ratio of DE pressure to its energy density.

However, recent observational studies suggest that current data lacks the precision to clearly distinguish among the possibilities $\omega < -1$, $\omega = -1$, and $\omega > -1$ [34–36]. During the decelerating expansion phases of the Universe, which include the radiation-dominated and matter-dominated epochs, the EoS parameter takes the values $\omega = \frac{1}{3}$ and $\omega = 0$ respectively. In contrast, the presently observed accelerating phase corresponds to $\omega < -\frac{1}{3}$. This condition encompasses three regimes: quintessence ($-1 < \omega < -\frac{1}{3}$), the cosmological constant case ($\omega = -1$, as in the ΛCDM model), and the phantom regime ($\omega < -1$).

Several major cosmological missions have reported observational constraints on the EoS parameter. The Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP9),which combined data from Hubble parameter measurements, supernovae (SNe), the cosmic microwave background (CMB), and baryon acoustic oscillations (BAO), estimated $\omega = -1.084 \pm 0.063$ [34]. The Planck collaboration later refined this to $\omega = -1.006 \pm 0.0451$ [35], and further updated the value in 2018 to $\omega = -1.028 \pm 0.032$ [36] . These results indicate a transition in the EoS parameter from positive values in the early Universe to negative values in the present epoch.

To explore the evolution of DE in a consistent framework, researchers often employ various parameterizations of the EoS. These models are useful for reconstructing the dynamics of the universe and constraining DE behavior. A number of such parameterizations are discussed in the literature [37–40]. In this study, we adopt the SR parameterization, where the EoS parameter is expressed as a function of redshift z. Accordingly, we denote the EoS of dark energy as $\omega_Q = \omega_{DE} = \omega_{SR}$ [41].

$$\omega_{\mathrm{SR}}(z) = \omega_0 + \omega_1 \frac{z}{\sqrt{z^2+1}}. \tag{3.23}$$

This form provides a direct physical interpretation ω_0 gives the present-day value of the EoS, and ω_1 governs its evolution. Moreover, this formulation accommodates various dark energy models by altering the values of ω_0 and ω_1. At early times ($z \to \infty$), the EoS parameter becomes

$$\omega_{\mathrm{SR}} = \omega_0 + \omega_1 \tag{3.24}$$

and at late times ($z \to -1$) it becomes

$$\omega_{\mathrm{SR}} = \omega_0 - \frac{\omega_1}{\sqrt{2}}. \tag{3.25}$$

Using equations (3.21), (3.22) and (3.23), we derive the differential equation,

$$\frac{dH(z)}{dz} = \frac{3\left(\omega_0 + \frac{\omega_1 z}{\sqrt{z^2+1}} + 1\right)}{2n(z+1)} H(z). \tag{3.26}$$

Solving this yields,

$$H(z) = c_0 \exp\left(\frac{3}{2n}\left[\frac{\omega_1 \tanh^{-1}\left(\frac{1-z}{\sqrt{2}\sqrt{z^2+1}}\right)}{\sqrt{2}} + (\omega_0+1)\log(z+1) + \omega_1 \sinh^{-1}(z)\right]\right), \tag{3.27}$$

where c_0 is an integration constant. Evaluating at $z = 0$, the present value of the Hubble parameter is given by

$$H_0 = c_0 \exp\left(\frac{3\omega_1 \coth^{-1}(\sqrt{2})}{2\sqrt{2}n}\right). \tag{3.28}$$

Rewriting $H(z)$ in terms of H_0, we get

$$H(z) = H_0(1+z)^{\frac{3(\omega_0+1)}{2n}} \exp\left(\frac{6\omega_1 \sinh^{-1}(z) - 3\sqrt{2}\omega_1 \left[\tanh^{-1}\left(\frac{z-1}{\sqrt{2}\sqrt{z^2+1}}\right) + \coth^{-1}(\sqrt{2})\right]}{4n} \right). \tag{3.29}$$

The deceleration parameter q is given by

$$q = -1 + \frac{(1+z)}{H(z)} \frac{dH(z)}{dz}, \tag{3.30}$$

which for our model, becomes

$$q(z) = -1 + \frac{1}{2n}\left(3\omega_0 + \frac{3\omega_1 z}{\sqrt{z^2+1}} + 3\right). \tag{3.31}$$

A negative value of $q(z)$ indicates acceleration, while a positive value indicates deceleration.

The $Om(z)$ diagnostic is useful in distinguishing different DE models [42], defined as

$$Om(z) = \frac{\left(\frac{H(z)}{H_0}\right)^2 - 1}{(1+z)^3 - 1}. \tag{3.32}$$

A negative slope of $Om(z)$ indicates quintessence-like behavior ($\omega > -1$), a positive slope implies phantom behavior ($\omega < -1$) and a constant slope corresponds to the ΛCDM model ($\omega = -1$).

3.4 Cosmological Parameters

Cosmological parameters are fundamental in shaping viable models of the universe. For a cosmological model to be considered realistic, its predicted present value of the deceleration parameter $q(z)$ must align with observational data. Moreover, the sign of q plays a crucial role in distinguishing between decelerated and accelerated phases of cosmic expansion a positive q indicates deceleration, while a negative q corresponds to acceleration. In the following, we present the behavior of key cosmological quantities, including the deceleration parameter, energy density, and EoS parameter, as functions of the redshift z.

Figure 3.1 illustrates the evolution of the deceleration parameter $q(z)$. The figure clearly exhibits

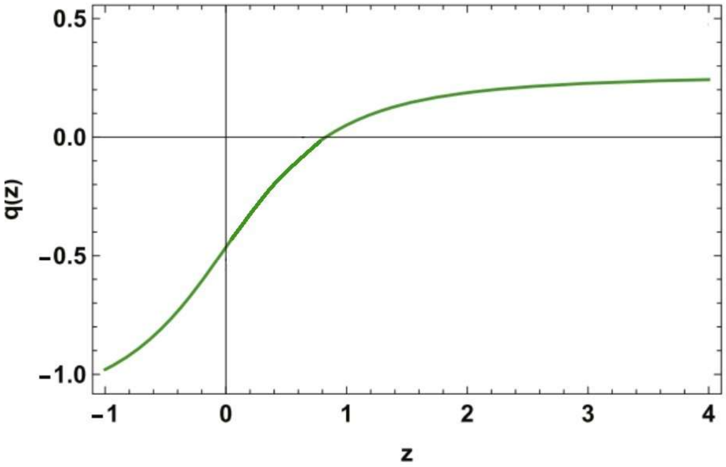


Figure 3.1: The behavior of the deceleration parameter (q) versus redshift (z).

a transition from a decelerating phase ($q > 0$) to an accelerating phase ($q < 0$), marking a shift in the universe's expansion behavior. The present-day value of the deceleration parameter (i.e., at $z = 0$) is found to be $q_0 = -0.46^{+0.07}_{-0.16}$. The redshift at which the transition occurs, often referred to as the transition redshift z_{tr} is approximately $z_{\mathrm{tr}} = 0.83^{+0.33}_{-0.54}$ [43, 44].

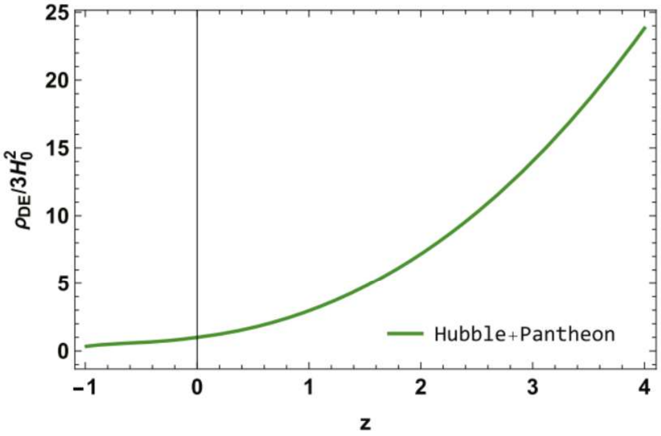


Figure 3.2: The behavior of the density parameter for ρ_{DE} versus redshift (z).

The variation of the dark energy density ρ_{DE} with redshift is shown in figure 3.2. As the redshift increases i.e., moving toward earlier times, the DE density also increases. Conversely, with the progression of cosmic time, ρ_{DE} gradually decreases and tends toward a minimum value at late times. The relatively small value of dark energy density at late epochs suggests that the universe is expected to continue accelerating, potentially culminating in a future big rip scenario.

The EoS parameter (ω) is a key quantity in characterizing the expansion state of the universe. When $\omega < -\frac{1}{3}$, the universe experiences accelerated expansion. Figure 3.3 illustrates the be-

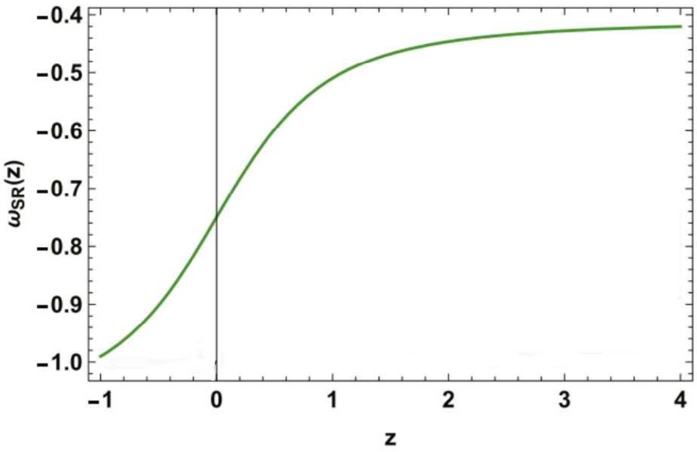


Figure 3.3: The behavior of the EoS parameter for DE (ω) versus redshift (z).

havior of the EoS parameter for dark energy. The plot demonstrates that the EoS remains in the quintessence regime throughout cosmic evolution, indicating that dark energy continues to drive the acceleration. The present value of the EoS parameter is estimated as $\omega_0 \approx -0.75$.

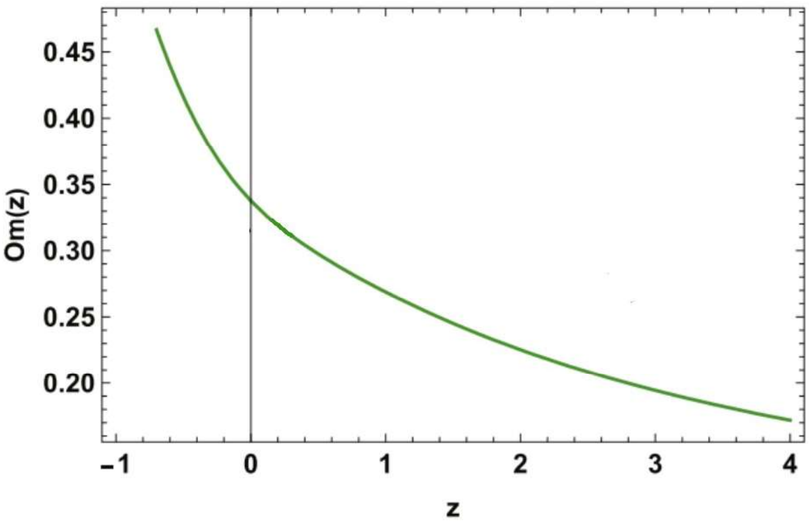


Figure 3.4: The behavior of the Om(z) versus redshift (z).

Additionally, the slope of the Om diagnostic function, $\mathrm{Om}(z)$ serves as a tool to distinguish between various dark energy scenarios, such as quintessence or phantom. As shown in figure 3.4, the Om diagnostic exhibits a negative slope across the redshift range considered. This behavior is consistent with a quintessence-like nature of dark energy in the current cosmological model.

3.5 Conclusion

Dark energy (DE) remains one of the most compelling topics in modern cosmology due to its significant role in driving the accelerated expansion of the universe. Numerous approaches have been proposed to account for this acceleration, including various DE parametrizations and modified gravitational theories. In this work, we investigated the SR parametrization within the framework of $f(Q)$ gravity a modified theory where gravitational effects are attributed to the non-metricity scalar Q, providing an alternative to GR. We obtained exact solutions of the field equations by assuming the functional form $f(Q) = Q + \alpha Q^n$, with α and n being arbitrary constants, and by incorporating the SR parametrization of the equation of state for DE as

$$\omega(z) = \omega_0 + \omega_1 \frac{z}{1+z^2},$$

which leads to a variable deceleration parameter.

To determine the best-fit values of the model parameters ω_0, ω_1, and n, we utilized the Hubble dataset consisting of 31 observational points along with the Pantheon compilation of 1048 supernovae data points. The combined analysis yields the following best-fit estimates: $H_0 = 67.83^{+0.75}_{-0.74} km/s/Mpc, \omega_0 = -0.75^{+0.19}_{-0.20}, \omega_1 = -0.34^{+0.30}_{-0.34}, n = 0.74^{+0.39}_{-0.26}$. Here, ω_0 represents the current value of the EoS parameter and corresponds to a negative value, placing it within the quintessence regime. We also explored several cosmological parameters to test the physical viability of the model. The deceleration parameter indicates a smooth transition from an early decelerating epoch to the current accelerated expansion.For the combined Hubble+Pantheon dataset, the present value of the deceleration parameter is $q_0 = -0.46^{+0.07}_{-0.16}$, and the redshift corresponding to the transition from deceleration to acceleration is $z_{\text{tr}} = 0.83^{+0.33}_{-0.54}$.

Furthermore, figure 3.4 illustrates that the Om(z) diagnostic function exhibits a negative slope, further confirming the quintessence-like behavior of our model. Based on these results, the considered $f(Q)$ cosmological model stands out as a viable and promising alternative to the standard ΛCDM model. A more comprehensive exploration of this framework will be pursued in future studies.

Bibliography

[1] T. Matos, F. S. Guzmán, L. A. Urena-López, Scalar field as dark matter in the universe, Classical and Quantum Gravity, 17, 1707, 2000, IOP Publishing.

[2] A. G. Riess et al., Observational evidence from supernovae for an accelerating universe and a cosmological constant, The Astronomical Journal, 116, 1009, 1998.

[3] S. Perlmutter et al., Measurements of Ω and Λ from 42 high-redshift supernovae, The Astrophysical Journal, 517, 565–586, 1999.

[4] D. J. Eisenstein et al., Detection of the Baryon Acoustic Peak in the Large-Scale Correlation Function of SDSS Luminous Red Galaxies, The Astrophysical Journal, 633, 560, 2005.

[5] P. J. E. Peebles, Principles of Physical Cosmology, Princeton University Press, 1993.

[6] Planck Collaboration, Planck 2018 results. VI. Cosmological parameters, Astronomy & Astrophysics, 641, A6, 2020.

[7] A. G. Riess et al., A 2.4% Determination of the Local Value of the Hubble Constant, The Astrophysical Journal, 826, 56, 2016.

[8] V. Sahni, T. D. Saini, A. A. Starobinsky, and U. Alam, Statefinder—a new geometrical diagnostic of dark energy, JETP Letters, 77, 201–206, 2003.

[9] J. Beltrán Jiménez, L. Heisenberg, and T. Koivisto, Coincident General Relativity, Phys. Rev. D 98, 044048,2018.

[10] R. Lazkoz, F. S. N. Lobo, M. Ortiz-Baños, and V. Salzano, Observational constraints of $f(Q)$ gravity, Phys. Rev. D 100, 104027, 2019.

[11] T. Harko, T. S. Koivisto, F. S. N. Lobo, and G. J. Olmo, Metric-Palatini gravity unifying local constraints and late-time cosmic acceleration, Phys. Rev. D 85, 084016, 2012.

[12] Y. Xu, G. Li, T. Harko, and S.-D. Liang, $f(Q,T)$ gravity, Eur. Phys. J. C 79, 708, 2019.

[13] W. Khyllep, A. Paliathanasis, and J. Dutta, Cosmological solutions and growth index of matter perturbations in $f(Q)$ gravity, Phys. Rev. D, 103, 103521, 2021.

[14] M. Mandal, A. Das, and T. Paul, Cosmography in $f(Q)$ gravity, Phys. Rev. D 102, 124029,2020.

[15] S. Capozziello, M. Capriolo, and L. Caso, Weak field limit and gravitational waves in $f(Q)$ gravity, Eur. Phys. J. C, 80, 156, 2020.

[16] R. Solanki, A. De, and P. K. Sahoo, *Cosmological dynamics and observational constraints in $f(Q)$ gravity*, Phys. Dark Univ. **32**, 100820 (2021).

[17] V. C. Busti et al., Model-independent constraints on $f(Q)$ gravity, Phys. Lett. B, 813, 136034, 2021.

[18] J. B. Jimenez, L. Heisenberg, T. Koivisto, Coincident general relativity, Physical Review D, 98, 044048, 2018, APS.

[19] L. Heisenberg, M. Hohmann,S. Kuhn,Homogeneous and isotropic cosmology in general teleparallel gravity, The European Physical Journal C, 83, 315, 2023, Springer.

[20] R. Lazkoz, F SN. Lobo, B. Ortiz, Observational constraints of f (Q) gravity, Physical Review D, 100, 104027,2019, APS.

[21] L. Amendola, M. Kunz, I. Saltas, Observables and unobservables in dark energy cosmologies, Physical Review D Particles, Fields, Gravitation, and Cosmology, 87, 023501, 2013, APS.

[22] S. K. J. Pacif, R. Myrzakulov, S. Myrzakul, Reconstruction of cosmic history from a simple parametrization of H, International Journal of Geometric Methods in Modern Physics, 14, 1750111, 2017, World Scientific.

[23] SH. Shekh, M. Muzammil, RV. Mapari, Exploring holographic dark energy with Hubble's and Granda–Oliveros horizons as the infrared cut-off in non-static plane symmetric spacetime, International Journal of Geometric Methods in Modern Physics, 20, 2350233, 2023, World Scientific.

[24] M. Koussour, SKJ. Pacif, M. Bennai, Pk. Sahoo, A New Parametrization of Hubble Parameter in f (Q) f(Q) Gravity, Fortschritte der Physik, 71, 2200172, 2023, Wiley Online Library.

[25] V. Sahni, Arman Shafieloo, Alexei A Starobinsky, Two new diagnostics of dark energy, Physical Review D, 78, 103502, 2008, APS.

[26] S. mandal, PK. sahoo, J. RL. santos, Energy conditions in f (Q) gravity, Physical Review D, 102, 024057, 2020, APS.

[27] A. De, s. Mandal, JT. Beh, T. Loo, Isotropization of locally rotationally symmetric Bianchi-I universe in f (Q)-gravity, The European Physical Journal C, 82, 72, 2022, Springer.

[28] W.Khyllep, A. Paliathanasis, J. Dutta, Cosmological solutions and growth index of matter perturbations in f (Q) gravity, Physical Review D, 103, 103521, 2021, APS.

[29] M. Koussour, N. Myrzakulov, HA. A. Alfedeel, EI. Hassan, Square-root parametrization of dark energy in f (Q) cosmology, Communications in Theoretical Physics, 75, 125403, 2023, IOP Publishing.

[30] R. Lazkoz, SN. Francisco, B. Ortiz, Observational constraints of f (Q) gravity, @article-lazkoz2019observational, Physical Review D, 100, 104027, 2019,APS.

[31] BJ. Barros, T. Barreiro, T. Koivisto, Testing F (Q) gravity with redshift space distortions, Physics of the Dark Universe, 30, 100616, 2020, Elsevier.

[32] JB. Jiménez, L. Heisenberg, T. Koivisto, Cosmology in f (Q) geometry, Physical Review D, 101, 103507, 2020, APS.

[33] W. Khyllep, A. Paliathanasis, J. Dutta, Cosmological solutions and growth index of matter perturbations in f (Q) gravity, Physical Review D, 103, 103521, 2021, APS.

[34] G. Hinshaw, D. Larson, E. Komatsu, S. Eiichiro, Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: cosmological parameter results, The Astrophysical Journal Supplement Series, 208, 19, 2013, IOP Publishing.

[35] AR. P. Ade, N. Aghanim, M. Arnaud, Planck 2015 results-xiii. cosmological parameters, Astronomy & Astrophysics, 594, A13, 2016, EDP sciences.

[36] N. Aghanim and others, Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys, 641, A6, 2020.

[37] EM. Barboza Jr,JS. Alcaniz, A parametric model for dark energy, Physics Letters B, 666, 415–419, 2008, Elsevier.

[38] R. C. Nunes, E.Barboza, Dark matter–dark energy interaction for a time-dependent EoS parameter, General Relativity and Gravitation, 46, 1–10, 2014, Springer.

[39] H. Wei, X. Yan, Y. Zhou, Cosmological applications of Pade approximant,Journal of Cosmology and Astroparticle Physics, 2014, 01, 2014, IOP Publishing.

[40] L. G. Jaime, M. Jaber, C. Escamilla-Rivera, New parametrized equation of state for dark energy surveys, Physical Review D, 98, 083530, 2018, Aps.

[41] G. Pantazis, S. Nesseris, L. Perivolaropoulos, Comparison of thawing and freezing dark energy parametrizations, Physical Review D, 93, 103503, 2016, APS.

[42] V. Sahni, A. Shafieloo, A. A. Starobinsky, Two new diagnostics of dark energy, Physical Review D—Particles, Fields, Gravitation, and Cosmology, 78, 103502, 2008, APS.

[43] A. Al Mamon, K. Bamba, Observational constraints on the jerk parameter with the data of the Hubble parameter, The European Physical Journal C, 78, 862, 2018, Springer.

[44] O.Farooq, F. R. Madiyar, S. Crandall, B. Ratra, Hubble parameter measurement constraints on the redshift of the deceleration–acceleration transition, dynamical dark energy, and space curvature, The Astrophysical Journal, 835, 26, 2017, IOP Publishing.